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ASYMPTOTIC REGRESSION WITH VECTOR PARAMETERS [ARVP]

(A computer based technique to fit growth response curves)

by

Donald L. Schwandt

MICHIGAN TECHNOLOGICAL UNIVERSITY
FORD FORESTRY CENTER
L'ANSE, MICHIGAN 49946

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ABSTRACT

A technique for estimating the vector parameters for asymptotic regression was developed. This technique was used to fit, simultaneously, the site index curves for red maple, sugar maple, aspen, and yellow birch. This procedure should prove beneficial for growth studies when working with a variety of sites, stocking levels, and species composition.

INTRODUCTION

Various techniques of regression analysis have been used to develop mathematical functions for use in modelling forest growth processes. Because these processes are seldom linear in nature many researchers have resorted to polynomial regression and logrithmic transformations to approximate the relationships. Though these techniques often provide reasonably adequate approximations, they occasionally fail to represent the true nature of the process being investigated. A few examples in which polynomial regression would be inappropriate are:

1. modelling stand periodic basal area growth rate in relation to initial stand basal area;
2. modelling stand periodic cubic foot (meter) volume growth rate in relation to initial stand volume or basal area; and,
3. relating total tree height to tree age.

In (1) and (2) avove, the growth rate is asymptotic with respect to stand stocking levels. Understocked stands grow at a low rate. As stocking increases the growth rate increases at a decreasing rate until full site utilization is achieved. At full stocking the growth rate stabilizes (with some variation) at the site potential growth rate. Likewise, the relationship between tree height and age can be viewed as asymptotic. Individual trees go through a period of rapid height development followed by a gradual reduction in height growth until total mature-height is attained. At this point height growth ceases.

As an alternative to polynomial regression many researchers have employed various forms of the asymptotic regression equation,

$$y = \alpha + \beta\rho^{z} \qquad (1)$$

to model growth processes and physical laws when the response variable, y, is asymptotic with respect to the independent variable, z. In its standard form it has been used by Stevens (1951) to model Newton's Law of Cooling and the Law of Diminishing Returns which relates crop response to fertilization rate (Mitscherlich's Law). It has been used in the field of economics and business to model inventory systems and to describe sales of certain consumer goods (Manetsch and Park, 1974; Jedamus, Frame and Taylor, 1976). This particular form has many additional applications, but a historical account is not the objective here.

Taking the reciprocal of y and rewriting equation (1) in the form,

$$1/y = \alpha + \beta\rho^{z} \tag{2}$$

yields the logistic function. This is the familiar S-shaped curve commonly used to model indicator variables. Probably the most common applications of the logistic function are in tree mortality studies, construction of actuary tables, and herbicide studies.

Yet another function that can be fit using equation (1) is the Chapman-Richards growth function. The fitting procedure requires two steps and the reader should refer to Richards (1959), Pienaar and Turnbull (1973), or Rawat and Franz (1974) for details. This particular function has been extensively used in recent years. Moser and Hall (1969) employed it to describe basal area growth of northern hardwoods. Pienaar and Turnbull used it to develop growth and yield functions, and site index curves for spruce and pine. Rawat and Franz used asymptotic regression to fit Chapman-Richards function to site index curves of a variety of species growing on a variety of sites. They then used multiple and polynomial regression to relate the model parameters of each tree to measures of site quality and species. In so doing, they constructed a family of site index curves for ten tree species.

Though the discussion of previous studies investigating asymptotic relationships is brief, it does serve to highlight two important points. First, asymptotic regression is an important tool for use in forest growth and yield studies. Second, the parameter estimation procedures are restricted to the case of scalar parameters. The latter is of primary concern in this note. For example, Rawat and Franz (1974) developed a set of site index curves by applying asymptotic regression, individually, to each tree's height-over-age curve. This required a separate estimate of scaler α, β, and ρ for each tree in the study, a time consuming task. Only after all trees were fit could the individual parameter estimates be related to site and species; and this required separate regression analysis for each parameter. A superior, time efficient, technique to generate the same set of curves would have been to simultaneously quantify the effect of site, species, and age on the height development of all trees in the study. This could be facilitated by replacing the scalar parameters α, β, and ρ with vectors.

The purpose of this note is to describe and demonstrate a technique for estimating the vector parameters of the asymptotic regression model,

$$y = \alpha + \beta\rho^{z}, \tag{3}$$

where:

y is the dependent variable. It is assumed to approach an asymptotic upper (or lower) limit as z approaches infinity.

α represents the asymptotic value of y. $\alpha = a_0X1_0 + a_1X1_1 + \ldots + a_jX1_j$, that is, α is itself determined by a linear combination of vector coefficients, a, and a vector of observations taken on j independent variables that affect the asymptotic value of y.

β represents the total change in y as z passes from 0 to $+\infty$. $\beta = b_0X2_0 + b_1X2_1 + \ldots + b_kX2_k$; β is determined by a linear combination of vector coefficients, b, and a vector of observations taken on k independent variables that affect the magnitude of this total change in y.

ρ represents the factor by which the rate of change in y reduces as y approaches its asymptote. $\rho = r_0X3_0 + r_1X3_1 + \ldots + r_mX3_m$.

z is the primary independent variable on which y is defined.

METHODS

The technique used to estimate the vector parameters for equation (3) is precisely that of Stevens (1951) with appropriate modifications to handle the expanded estimation space. In an attempt to conserve space only a brief description will be presented here. This particular procedure is a part of a larger, interactive, statistical package developed by the author. Details and copies of the entire package will be made available under a different cover.

As an aide in describing the technique, equation (3) can be rewritten as follows:

$$y = a_0X1_0 + a_1X1_1 + \ldots + a_jX1_j + (b_0X2_0 + b_1X2_1 + \ldots + b_kX2_k)(r_0X3_0 + r_1X3_1 + \ldots + r_mX3_m)^z \qquad (4)$$

Using the least squares objective function the normal equations are given in Table 1. Continuing to follow Stevens procedure, the information matrix, $\{I\}$, is found by differentiating the normal equations with respect to each element in the vector parameters, α, β, and ρ, and changing all signs (Table 2). Inverting the information matrix yields the covariance matrix, $\{V\}$. The estimation procedure utilizes the normal equations and the covariance matrix in a two-stage, iterative optimization routine.

The first stage is designed to compute an overall fit of equation (3) under the assumption of scalar parameters. Initiation of this stage requires certain information from the user. The user must provide a vector of observations of y and z. The rank and elements of the matrices $X1$, $X2$, and $X3$ must also be provided. Care must be taken to ensure that the first column vector in $X1$, $X2$, and $X3$ is a unit vector of order n. Finally, the user must provide an initial estimate of scalar ρ and a convergence parameter δ. The initial

Table 1. *Generalized set of normal equations for asymptotic regression with vector parameters.*

$$a_0\text{:}\quad -S(X1_0[N + ML^z - Y]) = 0$$
$$a_1\text{:}\quad -S(X1_1[N + ML^z - Y]) = 0$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$a_j\text{:}\quad -S(X1_j[N + ML^z - Y]) = 0$$
$$b_0\text{:}\quad -S(X2_0 L^z[N + ML^z - Y]) = 0$$
$$b_1\text{:}\quad -S(X2_1 L^z[N + ML^z - Y]) = 0$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$b_k\text{:}\quad -S(X2_k L^z[N + ML^z - Y]) = 0$$
$$r_0\text{:}\quad -S(zX3_0 L^{z-1} M[N + ML^z - Y]) = 0$$
$$r_1\text{:}\quad -S(zX3_1 L^{z-1} M[N + ML^z - Y]) = 0$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$r_1\text{:}\quad -S(zX3_m L^{z-1} M[N + ML^z - Y]) = 0$$

where: $L = r_0X3_0 + r_1X3_1 + \ldots + r_mX3_m$

$M = b_0X2_0 + b_1X2_1 + \ldots + b_kX2_k$

$N = a_0X1_0 + a_1X1_1 + \ldots + a_jX1_j$

$S(\ldots)$ defines the summation over all n observations.

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estimate of ρ may be estimated via techniques described by Stevens (1951) and Jedamus <u>et</u>. <u>al</u>. (1976). δ defines the maximum allowable change in the elements of vector, ρ, over five successive iterations, for convergence; this is usually set at a relatively small number, say $\delta \pm .0001$.

Given the foregoing information the system takes over generating estimates of scalar α, β, and ρ; or a_0, b_0, and r_0, given that $a_i = b_i$. $r_i = 0$ for $i \geq 1$ (note: during stage 1, X1, X2, and X3 are assumed to be (n x 1) unit matrices and are therefore dropped until stage 2). First, it takes the initial guess of r_0, constructs a vector, $\omega = r_0^z$, and calls a linear regression algorythm to arrive at starting values for a_0, and b_0 (i.e., solves the linear regression equation, $y = a_0 + b_0\omega$, for a_0 and b_0). Next the initial estimates of the scalars, a_0, b_0, and r_0, are entered in the left side of the normal equations yielding a vector, γ, whose rank is 3, and whose elements are not equal 0 because a_0, b_0, and r_0 are not efficient estimates of α, β, and ρ. Then, the initial parameter estimates are used to construct the information matrix, {I}, a(3 x 3) symetric matrix. Inverting {I} yields {V} and new estimates of a_0, b_0, and r_0 are calculated by solving the equation,

Table 2. *Generalized information matrix for asymptotic regression with vector parameters.*

$$
\{:\} = \left\{\begin{array}{cccc|cccc|cccc}
S(X1_0^2) & S(X1_0X1_1) & \cdots & S(X1_0X1_j) & S(X1_0X2_0L^z) & S(X1_0X2_1L^z) & \cdots & S(X1_0X2_kL^z) & S(zX1_0X3_0L^{z-1}M) & S(zX1_0X3_1L^{z-1}M) & \cdots & S(zX1_0X3_mL^{z-1}M) \\
S(X1_0X1_1) & S(X1_1^2) & \cdots & S(X1_1X1_j) & S(X1_1X2_0L^z) & S(X1_1X2_1L^z) & \cdots & S(X1_1X2_kL^z) & S(zX1_1X3_0L^{z-1}M) & S(zX1_1X3_1L^{z-1}M) & \cdots & S(zX1_1X3_mL^{z-1}M) \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
S(X1_0X1_j) & S(X1_1X1_j) & \cdots & S(X1_j^2) & S(X1_jX2_0L^z) & S(X1_jX2_1L^z) & \cdots & S(X1_jX2_kL^z) & S(zX1_jX3_0L^{z-1}M) & S(zX1_jX3_1L^{z-1}M) & \cdots & S(zX1_jX3_mL^{z-1}M) \\
\hline
S(X1_0X2_0L^z) & S(X1_1X2_0L^z) & \cdots & S(X1_jX2_0L^z) & S(X2_0^2L^{2z}) & S(X2_0X2_1L^{2z}) & \cdots & S(X2_0X2_kL^{2z}) & S(zX2_0X3_0L^{2z-1}M) & S(zX2_0X3_1L^{2z-1}M) & \cdots & S(zX2_0X3_mL^{2z-1}M) \\
S(X1_0X2_1L^z) & S(X1_1X2_1L^z) & \cdots & S(X1_jX2_1L^z) & S(X2_0X2_1L^{2z}) & S(X2_1^2L^{2z}) & \cdots & S(X2_1X2_kL^{2z}) & S(zX2_1X3_0L^{2z-1}M) & S(zX2_1X3_1L^{2z-1}M) & \cdots & S(zX2_1X3_mL^{2z-1}M) \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
S(X1_0X2_kL^z) & S(X1_1X2_kL^z) & \cdots & S(X1_jX2_kL^z) & S(X2_0X2_kL^{2z}) & S(X2_1X2_kL^{2z}) & \cdots & S(X2_k^2L^{2z}) & S(zX2_kX3_0L^{2z-1}M) & S(zX2_kX3_1L^{2z-1}M) & \cdots & S(zX2_kX3_mL^{2z-1}M) \\
\hline
S(zX1_0X3_0L^{z-1}M) & S(zX1_1X3_0L^{z-1}M) & \cdots & S(zX1_jX3_0L^{z-1}M) & S(zX2_0X3_0L^{2z-1}M) & S(zX2_1X3_0L^{2z-1}M) & \cdots & S(zX2_kX3_0L^{2z-1}M) & S(z^2X3_0^2L^{2z-2}M^2) & S(z^2X3_0X3_1L^{2z-2}M^2) & \cdots & S(z^2X3_0X3_mL^{2z-2}M^2) \\
S(zX1_0X3_1L^{z-1}M) & S(zX1_1X3_1L^{z-1}M) & \cdots & S(zX1_jX3_1L^{z-1}M) & S(zX2_0X3_1L^{2z-1}M) & S(zX2_1X3_1L^{2z-1}M) & \cdots & S(zX2_kX3_1L^{2z-1}M) & S(z^2X3_0X3_1L^{2z-2}M^2) & S(z^2X3_1^2L^{2z-2}M^2) & \cdots & S(z^2X3_1X3_mL^{2z-2}M^2) \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
S(zX1_0X3_mL^{z-1}M) & S(zX1_1X3_mL^{z-1}M) & \cdots & S(zX1_jX3_mL^{z-1}M) & S(zX2_0X3_mL^{2z-1}M) & S(zX2_1X3_mL^{2z-1}M) & \cdots & S(zX2_kX3_mL^{2z-1}M) & S(z^2X3_0X3_mL^{2z-2}M^2) & S(z^2X3_1X3_mL^{2z-2}M^2) & \cdots & S(z^2X3_m^2L^{2z-2}M^2)
\end{array}\right\}
$$

where: M and L are defined in Table 1.

$$\begin{pmatrix} a_0 \\ b_0 \\ r_0 \end{pmatrix} = \begin{pmatrix} a_0' \\ b_0' \\ r_0' \end{pmatrix} + V\gamma,$$

where $\begin{pmatrix} a_0' \\ b_0' \\ r_0' \end{pmatrix}$ is the vector of old, inefficient estimates. These new estimates are entered in the left hand side of the normal equations and the whole cycle is repeated sequentially until $|r_0 - r_0'| \leq \delta$ for five successive iterations.

This completes stage 1 and the system prints:

1. final estimates of scalar α, β, and ρ,
2. R-square value,
3. mean square error,
4. sum of the deviations,
5. covariance matrix.

The second stage is identical to the first except here the work is with the expanded vectors α, β, and ρ, the $(n \times j)$ matrix X1, $(n \times k)$ matrix X2, $(n \times m)$ matrix X3, the normal equations, and information matrix. For the first iteration a_0, b_0, and r_0 are given the values computed in stage 1. All remaining elements in a, b, and r, are set at zero. That is:

$$a = \begin{pmatrix} a_0 = a_0 \\ a_1 = 0 \\ \cdot \\ \cdot \\ \cdot \\ a_j = 0 \end{pmatrix}, \quad b = \begin{pmatrix} b_0 = b_0 \\ b_1 = 0 \\ \cdot \\ \cdot \\ \cdot \\ b_k = 0 \end{pmatrix}, \quad r = \begin{pmatrix} r_0 = r_0 \\ r_1 = 0 \\ \cdot \\ \cdot \\ \cdot \\ r_m = 0 \end{pmatrix},$$

for the first iteration. These initial values of a, b, and r are entered in the left-hand-side of the normal equations yielding γ whose rank is now, $R = j + k + m$. The $(R \times R)$ information matrix, $\{R\}$, is constructed and new estimates of the vector parameters are calculated:

$$\begin{pmatrix} a_0 \\ a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_j \\ b_0 \\ b_1 \\ \cdot \\ \cdot \\ \cdot \\ b_k \\ r_0 \\ r_1 \\ \cdot \\ \cdot \\ \cdot \\ r_m \end{pmatrix} = \begin{pmatrix} a_0' \\ a_1' \\ \cdot \\ \cdot \\ \cdot \\ a_j' \\ b_0' \\ b_1' \\ \cdot \\ \cdot \\ \cdot \\ b_k' \\ r_0' \\ r_1' \\ \cdot \\ \cdot \\ \cdot \\ r_m' \end{pmatrix} + I^{-1}\gamma.$$

As in stage 1, this cycle is repeated until the change in all elements of vector, r, is less than δ for five successive iterations. Once convergence is attained the system prints the vector parameters plus the additional information listed after stage 1.

EXAMPLE

The fit of site index curves for red maple, sugar maple, aspen, and yellow birch was used to demonstrate this fitting technique. Data consisted of digitized site-index values, read at ten-year increments, from Carmean's published site index curves (1978).

Data input to the asymptotic regression package was as follows:

1. The dependent variable, y, was total tree height, in feet, as read from Carmean's curves.

2. The primary independent variable, z, consisted of tree age, in years, divided by ten, corresponding to each height reading.

3. The matrices X1, X2, and X3: for this particular run X1 = X2 = X3, (n x 5) matrices, where n is the number of data points in y and z (n = 299). The first column vector, $X1_1$, was site index divided by ten. The remaining vectors are defined:

$$X1_2 = \begin{cases} \text{(site index)/10, if red maple} \\ 0\text{, otherwise} \end{cases}$$

$$X1_3 = \begin{cases} \text{(site index)/10, if sugar maple} \\ 0\text{, otherwise} \end{cases}$$

$$X1_4 = \begin{cases} \text{(site index)/10, if aspen} \\ 0\text{, otherwise} \end{cases}$$

4. The initial guess at the overall value of ρ was 0.82.

5. The convergence parameter was 0.001.

The results of stage 1 and stage 2 are given in tables 3 and 4, respectively. The covariance matrix was intentionally left out because its discussion goes beyond the intent of this note. The important feature is that Carmean's curves have been duplicated in a single run yielding a mathematical expression fitting four species over site classes from site index 40 to 85.

DISCUSSION

The demonstrated fit of Carmean's site index curves is only one example of the potential use of ARVP. This routine is currently being used to evaluate the contribution of secondary information sources on forest productivity prediction. One example of this type is the investigation of the relationship between soil mapping units and forest productivity. ARVP lends itself well to this kind of study because the effect of stocking levels, species composition, and number of stems can be eliminated. Then the effect of soil can be quantified in much the same manner as employed in multiple regression. If successful, ARVP can be an important tool in providing much needed interpretive information to supplement soil survey maps.

Table 3. *Effects of stage 1, the overall fit of asymptotic regression on height-over-age, all species and sites combined.*

Equation	$y = a_0 + b_0 r_0{}^z$
Estimates	$a_0 = 104.365$
	$b_0 = -108.089$
	$r_0 = 0.830310$
Fit	$R^2 = 0.684$
	MSE = 157.7201

Table 4. *Results of stage 2 - the final equation.*

Equation	$y = \overset{4}{\underset{i=0}{S}} a_i X1_i + (\overset{4}{\underset{i=0}{S}} b_i X2_i)(\overset{4}{\underset{i=0}{S}} r_i X3_i)^z$
Estimates	$a_0 = 59.7082$
	$a_1 = 6.35763$
	$a_2 = 5.35180$
	$a_3 = .808576$
	$a_4 = -.824059$
	$b_0 = -62.7856$
	$b_1 = -6.79100$
	$b_2 = -4.55680$
	$b_3 = -1.09150$
	$b_4 = .111580$
	$r_0 = .956614$
	$r_1 = -.0233141$
	$r_2 = .0111895$
	$r_3 = .00157139$
	$r_4 = -.00418807$
Fit	$R^2 = .9979$
	MSE = 1.0685
Iterations	11

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